Conditions for equality between entanglement-assisted and unassisted classical capacities of a quantum channel

M.E. Shirokov

Steklov Mathematical Institute, RAS, Moscow

msh@mi.ras.ru

Abstract

Several relations between the Holevo capacity and the entanglement-assisted classical capacity of a quantum channel are proved, necessary and sufficient conditions for their coincidence are obtained. In particular, it is shown that these capacities coincide if (correspondingly, only if) the channel (correspondingly, the χ–essential part of the channel) belongs to the class of classical-quantum channels (the χ-essential part is a restriction of a channel obtained by discarding all states useless for transmission of classical information). The obtained conditions and their corollaries are generalized to channels with linear constraints. By using these conditions it is shown that the question of coincidence of the Holevo capacity and the entanglement-assisted classical capacity depends on the constraint (even for classical-quantum channels).

Properties of the difference between the quantum mutual information and the χ-function (constrained Holevo capacity) of a quantum channel are explored.

1   Introduction

Informational properties of a quantum channel are characterized by a number of different capacities defined by type of transmitted information, by additional resources used to increase the rate of this transmission, by security requirements, etc.

Central roles in analysis of transmission of classical information through a quantum channel Φ are played by the Holevo capacity $\bar{C}(\Phi)$, the classical (unassisted) capacity $C(\Phi)$ and the entanglement-assisted (classical) capacity $C_{\mathrm{ea}}(\Phi)$ of this channel. The first of them is defined as the maximal rate of information transmission between transmitter and receiver (generally called Alice and Bob) when nonentangled block coding is used by Alice and arbitrary measurement is used by Bob, the second one differs form the first by possibility to use arbitrary block coding by Alice while the entanglement-assisted capacity is defined as the maximal rate of information transmission between Alice and Bob under the assumption that they share a common entangled state, which can be used in block coding by Alice to increase the rate of information transmission [2, 16].

By the operational definitions $\bar{C}(\Phi) \leq C(\Phi) \leq C_{\mathrm{ea}}(\Phi)$. During a long time it was conjectured that $\bar{C}(\Phi) = C(\Phi)$ for any channel Φ until Hastings showed existence of a counter-example to the additivity conjecture [7]. Nevertheless, the equality $\bar{C}(\Phi) = C(\Phi)$ holds for a large class of channels including the noiseless channel, all unital qubit channels, all entanglement-breaking channels and many other concrete examples. In contract to this, possibility of the strict inequality $C(\Phi) < C_{\mathrm{ea}}(\Phi)$ was initially obvious, since the superdense coding implies that $C_{\mathrm{ea}}(\Phi) = 2C(\Phi) > 0$ if Φ is the noiseless channel. But there exist channels, for which

$$\bar{C}(\Phi) = C(\Phi) = C_{\mathrm{ea}}(\Phi) > 0 \tag{1}$$

(as an example one can consider the channel $\rho \mapsto \sum_k \langle k|\rho|k\rangle |k\rangle\langle k|$, where $\{|k\rangle\}$ is an orthonormal basis). Hence the question "How can the class of channels for which (1) holds be characterized?" naturally arises. In contrast to an intuitive point of view this class does not coincide with the class of entanglement-breaking channels: despite the fact that these channels annihilate entanglement of any state shared by Alice and Bob, their entanglement-assisted capacity may be greater then the classical unassisted capacity [2]. On the other hand, in [3] an example of non-entanglement-breaking channel for which $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$ is described (see Example 2 in Section 2.3 below). A step in finding answer to the above question was recently made in [9], where a criterion of (1) for the class of q-c channels defined by quantum observables is obtained.

In this paper some relations between the capacities $\bar{C}(\Phi)$ and $C_{\mathrm{ea}}(\Phi)$ as well as necessary and sufficient conditions for the equality $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$ are

obtained (Proposition 1, Theorems 1 and 2). In particular, it is shown that the equality $\bar{C}(\Phi) = C_{ea}(\Phi)$ holds if (correspondingly, only if) the channel Φ (correspondingly, the χ–essential part of the channel Φ) belongs to the class of classical-quantum channels (the χ-essential part is defined as a restriction of a channel to the set of states supported by the minimal subspace containing elements of *all* ensembles optimal for this channel in the sense of the Holevo capacity, see Definition 1).

Since in dealing with infinite dimensional channels it is necessary to impose particular constraints on the choice of input code-states, we also consider conditions for coincidence of the entanglement-assisted capacity with the Holevo capacity for quantum channels with linear constraints (Propositions 4 and 5). By using these conditions it is shown that even in the case of classical-quantum channels the question of coincidence of the above capacities depends on the form of the constraint (Example 3, Proposition 6).

In Section 4 properties of the difference between the quantum mutual information and the χ-function (the constrained Holevo capacity) of a quantum channel (considered as a function of an input state) are studied (Theorem 3). In particular, the sense of the maximal value of this function as a parameter characterizing "noise level" of a quantum channel is shown.

2 Unconstrained channels

Let $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_E$ be finite dimensional Hilbert spaces. In what follows $\Phi \colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ is a quantum channel and $\widehat{\Phi} \colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_E)$ is its complementary channel, defined uniquely up to unitary equivalence [12].[1]

Let $H(\rho)$ and $H(\rho\|\sigma)$ be respectively the von Neumann entropy of the state ρ and the quantum relative entropy of the states ρ and σ [16].

The Holevo capacity of the channel Φ can be defined as follows

$$\bar{C}(\Phi) = \max_{\rho \in \mathfrak{S}(\mathcal{H}_A)} \chi_\Phi(\rho), \tag{2}$$

where

$$\chi_\Phi(\rho) = \max_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\rho)) \tag{3}$$

is the χ-function of the channel Φ [13]. Note that

$$\chi_\Phi(\rho) = H(\Phi(\rho)) - \hat{H}_\Phi(\rho), \tag{4}$$

---

[1]The quantum channel $\widehat{\Phi}$ is also called *conjugate* to the channel Φ [15].

where $\hat{H}_\Phi(\rho) = \min_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H(\Phi(\rho_i))$ is the convex hull of the function $\rho \mapsto H(\Phi(\rho))$. By concavity of this function the above minimum can be taken over ensembles of pure states. An ensemble $\{\pi_i, \rho_i\}$ of pure states called *optimal for the channel* Φ if (cf. [17])

$$\bar{C}(\Phi) = \chi_\Phi(\bar{\rho}) = \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\bar{\rho})), \quad \bar{\rho} = \sum_i \pi_i \rho_i.$$

By the Holevo-Schumacher-Westmoreland theorem the classical capacity of the channel Φ can be expressed by the following regularization formula

$$C(\Phi) = \lim_{n \to +\infty} n^{-1} \bar{C}(\Phi^{\otimes n}).$$

By the Bennett-Shor-Smolin-Thapliyal theorem the entanglement-assisted capacity of the channel Φ is determined as follows

$$C_{\text{ea}}(\Phi) = \max_{\rho \in \mathfrak{S}(\mathcal{H}_A)} I(\rho, \Phi), \tag{5}$$

where $I(\rho, \Phi) = H(\rho) + H(\Phi(\rho)) - H(\widehat{\Phi}(\rho))$ is the quantum mutual information of the channel Φ at the state ρ [16].

By the operational definitions $\bar{C}(\Phi) \leq C(\Phi) \leq C_{\text{ea}}(\Phi)$. Analytically this follows (by means of (2) and (5)) from the following expression for the quantum mutual information:

$$I(\rho, \Phi) = H(\rho) + \chi_\Phi(\rho) - \chi_{\widehat{\Phi}}(\rho) = \chi_\Phi(\rho) + \Delta_\Phi(\rho), \tag{6}$$

where $\Delta_\Phi(\rho) = H(\rho) - \chi_{\widehat{\Phi}}(\rho)$. This expression is easily derived by using (4) and by noting that $\hat{H}_\Phi \equiv \hat{H}_{\widehat{\Phi}}$ (this follows from coincidence of the functions $\rho \mapsto H(\Phi(\rho))$ and $\rho \mapsto H(\widehat{\Phi}(\rho))$ on the set of pure states).

Since $H(\rho) = \sum_i \pi_i H(\rho_i \| \rho)$ for any ensemble $\{\pi_i, \rho_i\}$ of pure states with the average state ρ, we have

$$\Delta_\Phi(\rho) = \min_{\substack{\sum_i \pi_i \rho_i = \rho \\ \text{rank} \rho_i = 1}} \sum_i \pi_i \left[H(\rho_i \| \rho) - H(\widehat{\Phi}(\rho_i) \| \widehat{\Phi}(\rho)) \right] \geq 0, \tag{7}$$

where the last inequality follows from monotonicity of the relative entropy.

Remark 1. The minimum in (7) is achieved at an ensemble $\{\pi_i, \rho_i\}$ of pure states if and only if the maximum in (3) is achieved at this ensemble. Indeed, since $\sum_i \pi_i H(\Phi(\rho_i)) = \sum_i \pi_i H(\widehat{\Phi}(\rho_i))$, this can be easily shown by using expression (4) for the χ-functions of the channels Φ and $\widehat{\Phi}$.

2.1 General inequalities

Expression (6) immediately implies the general upper bound

$$C_{\mathrm{ea}}(\Phi) \leq \bar{C}(\Phi) + \log \dim \mathcal{H}_A,$$

proved in [5, 11] by different methods. By using this expression and by noting that $\chi_\Phi(\rho) - \chi_{\widehat{\Phi}}(\rho) = I_c(\rho, \Phi)$ is the coherent information of the channel Φ at the state ρ (see [18]) it easy to obtain the following inequalities[2]:

$$H(\rho_1) - \bar{C}(\widehat{\Phi}) \leq C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi)$$

$$\leq H(\rho_2) - \chi_{\widehat{\Phi}}(\rho_2) \underset{H(\Phi(\cdot)) \geq H(\cdot)}{\leq} H(\Phi(\rho_2)) - \chi_{\widehat{\Phi}}(\rho_2) = I_c(\rho_2, \Phi) + \hat{H}_\Phi(\rho_2),$$

(8)

where ρ_1 and ρ_2 are states in $\mathfrak{S}(\mathcal{H}_A)$ such that $\chi_\Phi(\rho_1) = \bar{C}(\Phi)$ (i.e. ρ_1 is the average state of an optimal ensemble) and $I(\rho_2, \Phi) = C_{\mathrm{ea}}(\Phi)$.

Let $Q_1(\Phi) = \max_{\rho \in \mathfrak{S}(\mathcal{H}_A)} I_c(\rho, \Phi)$ and $Q(\Phi) = \lim_{n \to +\infty} n^{-1} Q_1(\Phi^{\otimes n})$ be the quantum capacity of the channel Φ [16]. The following proposition contains several estimations derived from (8).

Proposition 1. *Let* $\Phi \colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *be a quantum channel and* $\widehat{\Phi} \colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_E)$ *its complementary channel.*

A) *The following inequalities hold*

$$\bar{C}(\Phi) - \bar{C}(\widehat{\Phi}) \leq C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) \underset{H(\Phi(\cdot)) \geq H(\cdot)}{\leq} Q_1(\Phi) + \min \sum_i \pi_i H(\Phi(\rho_i)), \quad (9)$$

$$C(\Phi) - C(\widehat{\Phi}) \leq C_{\mathrm{ea}}(\Phi) - C(\Phi) \underset{H(\Phi(\cdot)) \geq H(\cdot)}{\leq} Q(\Phi) + \min \sum_i \pi_i H(\Phi(\rho_i)), \quad (10)$$

where the minimum is over all ensembles $\{\pi_i, \rho_i\}$ *of pure states such that* $I\left(\sum_i \pi_i \rho_i, \Phi\right) = C_{\mathrm{ea}}(\Phi)$. *This term can be replaced by* $\max_{\rho \in \mathrm{extr}\, \mathfrak{S}(\mathcal{H}_A)} H(\Phi(\rho))$.

B) *If the average state of at least one optimal ensemble for the channel* Φ *coincides with the chaotic state* $\rho_c = (\dim \mathcal{H}_A)^{-1} I_A$ *then*

$$C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) \geq \log \dim \mathcal{H}_A - \bar{C}(\widehat{\Phi})$$

and hence $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi) \Rightarrow \bar{C}(\widehat{\Phi}) = \log \dim \mathcal{H}_A$.[3]

---

[2]Here and in what follows the subscription in the third inequality means that it holds under the condition $H(\Phi(\rho)) \geq H(\rho)$ for all $\rho \in \mathfrak{S}(\mathcal{H}_A)$. This condition is valid, in particular, for all bistochastic channels.

[3]Note that $\bar{C}(\widehat{\Phi}) \leq \log \dim \mathcal{H}_A$ for any channel Φ.

C) If $C_{\mathrm{ea}}(\Phi) = I(\rho_c, \Phi)$ then $\bar{C}(\widehat{\Phi}) = \log \dim \mathcal{H}_A \;\Rightarrow\; \bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$. If, in addition, the average state of at least one optimal ensemble for the channel $\widehat{\Phi}$ coincides with the chaotic state ρ_c then

$$C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) \leq \log \dim \mathcal{H}_A - \bar{C}(\widehat{\Phi}).$$

Proof. A) Inequality (9) directly follows from (8). To obtain inequality (10) by regularization from (8) it is sufficient to note that the function $\mathfrak{S}(\mathcal{H}_A^{\otimes n}) \ni \omega \mapsto I(\omega, \Phi^{\otimes n})$ attains maximum at the state $\rho_2^{\otimes n}$ by subadditivity of the quantum mutual information and to use the obvious inequality $\hat{H}_{\Phi^{\otimes n}}(\rho_2^{\otimes n}) \leq n \hat{H}_\Phi(\rho_2)$.

B) This assertion directly follows from inequality (8).

C) To derive the first part of this assertion from inequality (8) note that $\bar{C}(\widehat{\Phi}) = \log \dim \mathcal{H}_A$ implies $\bar{C}(\widehat{\Phi}) = \chi_{\widehat{\Phi}}(\rho_c)$. The second part directly follows from the second inequality in (8). $\qquad\square$

Remark 2. Since $\bar{C}(\widehat{\Phi}) \leq \log \dim \mathcal{H}_E$, we have

$$C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) \geq \log \dim \mathcal{H}_A - \log \dim \mathcal{H}_E$$

for any channel Φ satisfying the condition of Proposition 1, B) and hence $C_{\mathrm{ea}}(\Phi) > \bar{C}(\Phi)$ if the dimension of the environment (=the minimal number of Kraus operators) is less than the dimension of the input space of the channel Φ.

For an arbitrary channel Φ inequality (8) implies

$$C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) \geq H(\bar{\rho}) - \log \dim \mathcal{H}_E \geq \bar{C}(\Phi) - \log \dim \mathcal{H}_E,$$

where $\bar{\rho}$ is the average state of any optimal ensemble for the channel Φ.

2.2   Conditions for the equality $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$ based on the Petz theorem

By using expressions (6) and (7), monotonicity of the relative entropy and the Petz theorem [8, Theorem 3] characterizing the case in which monotonicity of the relative entropy holds with an equality, the following necessary and sufficient conditions for the equality $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$ can be obtained.

Theorem 1. *Let* $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *be a quantum channel and* $\widehat{\Phi}\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_E)$ *its complementary channel.*

A) *If there exist a channel* $\Theta\colon \mathfrak{S}(\mathcal{H}_E) \to \mathfrak{S}(\mathcal{H}_A)$ *and an ensemble* $\{\pi_i, \rho_i\}$ *of pure states such that*

$$\Theta(\widehat{\Phi}(\rho_i)) = \rho_i, \quad \forall i, \tag{11}$$

and $I(\bar{\rho}, \Phi) = C_{\mathrm{ea}}(\Phi)$, *where* $\bar{\rho} = \sum_i \pi_i \rho_i$, *then* $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$.[4]

B) *If* $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$ *then for an arbitrary optimal ensemble* $\{\pi_i, \rho_i\}$ *of pure states for the channel* Φ *with the average state* $\bar{\rho}$ *there exists a channel* $\Theta\colon \mathfrak{S}(\mathcal{H}_E) \to \mathfrak{S}(\mathcal{H}_A)$ *such that* (11) *holds. The channel* Θ *can be defined by means of an arbitrary non-degenerate probability distribution* $\{\hat{\pi}_i\}$ *by setting its action on any state* σ *supported by the subspace* $\mathrm{supp}\widehat{\Phi}(\bar{\rho})$ *as follows*

$$\Theta(\sigma) = [\hat{\rho}]^{1/2} \widehat{\Phi}^* \left(\left[\widehat{\Phi}(\hat{\rho})\right]^{-1/2} \sigma \left[\widehat{\Phi}(\hat{\rho})\right]^{-1/2} \right) [\hat{\rho}]^{1/2}, \tag{12}$$

where $\hat{\rho} = \sum_i \hat{\pi}_i \rho_i$ *and* $\widehat{\Phi}^*$ *is a dual map to the channel* $\widehat{\Phi}$.

If $\{\hat{\pi}_i\}$ *is a degenerate probability distribution then relation* (11) *holds for the channel* Θ *defined by* (12) *for all i such that* $\hat{\pi}_i > 0$.

Proof. A) If $\{\pi_i, \rho_i\}$ is an ensemble of pure states with the average state $\bar{\rho}$ for which (11) holds then monotonicity of the relative entropy and (7) imply $\Delta_\Phi(\bar{\rho}) = 0$ and hence $C_{\mathrm{ea}}(\Phi) = I(\bar{\rho}, \Phi) = \chi_\Phi(\bar{\rho}) \leq \bar{C}(\Phi)$.

B) Since $\chi_\Phi(\rho) \leq I(\rho, \Phi)$ for any state ρ by (6), it is easy to see that $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$ implies $\chi_\Phi(\bar{\rho}) = I(\bar{\rho}, \Phi)$ for any an optimal ensemble $\{\pi_i, \rho_i\}$ of pure states with the average state $\bar{\rho}$. It follows from (7) and Remark 1 that

$$H(\rho_i \| \bar{\rho}) = H(\widehat{\Phi}(\rho_i) \| \widehat{\Phi}(\bar{\rho})), \quad \forall i.$$

Hence the Petz theorem [8, Theorem 3] implies existence of the channel Θ for which (11) holds. By monotonicity of the relative entropy for arbitrary probability distribution $\{\hat{\pi}_i\}$ we have

$$H(\rho_i \| \hat{\rho}) = H(\widehat{\Phi}(\rho_i) \| \widehat{\Phi}(\hat{\rho})), \quad \hat{\rho} = \sum_i \hat{\pi}_i \rho_i,$$

for all i such that $\hat{\pi}_i > 0$. Hence the formula for the channel Θ also follows from the Petz theorem. $\square$

---

[4]It is sufficient to require that Θ is a trace preserving positive map for which monotonicity of the relative entropy holds.

Theorem 1, A) makes it possible to prove the equality $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$ for all classical-quantum channels (see Theorem 2 in Section 2.3).

Theorem 1, B) can be used to prove the strict inequality $C_{\text{ea}}(\Phi) > \bar{C}(\Phi)$, by showing that (11) can not be valid for an optimal ensemble $\{\pi_i, \rho_i\}$ and the channel Θ defined by (12).

Example 1. Consider the entanglement-breaking channel

$$\Phi(\rho) = \sum_k \langle \varphi_k | \rho | \varphi_k \rangle |k\rangle\langle k|,$$

where $\{|\varphi_k\rangle\}$ is an overcomplete system of vectors in the space $\mathcal{H}_A$ (that is $\sum_k |\varphi_k\rangle\langle\varphi_k| = I_A$) and $\{|k\rangle\}$ is an orthonormal basis in the space $\mathcal{H}_B$. It is easy to see that $\Phi = \widehat{\Phi}$. Hence $I(\rho, \Phi) = H(\rho)$ and $C_{\text{ea}}(\Phi) = \log \dim \mathcal{H}_A$.

Suppose that $\bar{C}(\Phi) = C_{\text{ea}}(\Phi) = \log \dim \mathcal{H}_A$. Then the average state of any optimal ensemble $\{\pi_i, \rho_i\}$ for the channel Φ coincides with the chaotic state ρ_c in $\mathfrak{S}(\mathcal{H}_A)$. Since $\widehat{\Phi}^*(A) = \sum_k \langle k|A|k\rangle|\varphi_k\rangle\langle\varphi_k|$ and $\widehat{\Phi}(\rho_c) = \Phi(\rho_c)$ is a full rank state, relation (11) can be valid for the channel Θ defined by (12) only if $\rho_i = |\varphi_{k_i}\rangle\langle\varphi_{k_i}|$ for some k_i and

$$\text{rank}\,\widehat{\Phi}(|\varphi_{k_i}\rangle\langle\varphi_{k_i}|) = \text{rank} \sum_k \langle\varphi_k|\varphi_{k_i}\rangle\langle\varphi_{k_i}|\varphi_k\rangle|k\rangle\langle k| = 1$$

for all i. But this can be valid only if $\{|\varphi_k\rangle\}$ is an orthonormal basis. So, we conclude that

$$C_{\text{ea}}(\Phi) = \bar{C}(\Phi) \quad \Leftrightarrow \quad \{|\varphi_k\rangle\} \text{ is an ortonormal basis.}$$

The same conclusion was obtained in [9] as a corollary of a general criterion for the equality $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$ for the class of channels defined by quantum observables, which is proved by means of the ensemble-measurement duality.

2.3 A simple criterion for the equality $\bar{C}(\Phi) = C_{\text{ea}}(\Phi)$.

Now we will show that the equality $\bar{C}(\Phi) = C_{\text{ea}}(\Phi)$ holds if (correspondingly, only if) the channel Φ (correspondingly, the subchannel of Φ determining its classical capacity) belongs to the class of classical-quantum channels.

A channel $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ is called *classical-quantum* if it has the following representation

$$\Phi(\rho) = \sum_{k=1}^{\dim \mathcal{H}_A} \langle k|\rho|k\rangle \sigma_k, \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \tag{13}$$

where $\{|k\rangle\}$ is an orthonormal basis in $\mathcal{H}_A$ and $\{\sigma_k\}$ is a collection of states in $\mathfrak{S}(\mathcal{H}_B)$ [14, 16].

For correct formulation of the above statement we will need the following notion.

Definition 1. Let $\mathcal{H}_\Phi^\chi$ be the minimal subspace of $\mathcal{H}_A$ containing elements of all optimal ensembles for the channel $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$. The restriction Φ_χ of the channel Φ to the set $\mathfrak{S}(\mathcal{H}_\Phi^\chi)$ is called *χ-essential* part (subchannel) of the channel Φ.

If $\mathcal{H}_\Phi^\chi \neq \mathcal{H}_A$ then pure states corresponding to vectors in $\mathcal{H}_A \setminus \mathcal{H}_\Phi^\chi$ can not be used as elements of optimal ensemble for the channel Φ. This means, roughly speaking, that these states are useless for non-entangled coding of classical information and hence it is natural to consider the χ-essential subchannel Φ_χ instead of the channel Φ dealing with the Holevo capacity of the channel Φ (which coincides with the classical capacity if $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$).

By definition $\bar{C}(\Phi_\chi) = \bar{C}(\Phi)$. Hence $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$ implies $C_{\mathrm{ea}}(\Phi_\chi) = C_{\mathrm{ea}}(\Phi)$. Thus, in this case speaking about the entanglement-assisted capacity of the channel Φ we may also consider the χ-essential subchannel Φ_χ instead of the channel Φ.

Theorem 1 makes it possible to prove the following assertions.

Theorem 2. *Let* $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *be a quantum channel.*

A) *If* Φ *is a classical-quantum channel then* $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$.

B) *If* $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$ *then the χ-essential part of the channel* Φ *is a classical-quantum channel.*

Example 2 below shows that in general the χ-essential part of the channel Φ in Theorem 2, B) can not replaced by the channel Φ.

Proof. A) If the channel Φ has representation (13) then $\Phi = \Phi \circ \Pi$, where $\Pi(\rho) = \sum_k \langle k|\rho|k\rangle |k\rangle\langle k|$ is a channel from $\mathfrak{S}(\mathcal{H}_A)$ to itself.

It is easy to show (see [4, the proof of Lemma 17]) existence of a channel Θ such that $\Theta \circ \widehat{\Phi \circ \Pi} = \widehat{\Pi} = \Pi$.

By the chain rule for the quantum mutual information (see [16]) we have

$$I(\rho, \Phi) = I(\rho, \Phi \circ \Pi) \leq I(\Pi(\rho), \Phi).$$

It follows that the function $\rho \mapsto I(\rho, \Phi)$ attains maximum at a state diagonalizable in the basis $\{|k\rangle\}$. Since $\Theta \circ \widehat{\Phi \circ \Pi}(|k\rangle\langle k|) = \Pi(|k\rangle\langle k|) = |k\rangle\langle k|$ for any k, Theorem 1, A) implies $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$.

B) Replacing the channel Φ by its χ-essential subchannel, we may consider that $\mathcal{H}_\Phi^\chi = \mathcal{H}_A$.

Let $\Phi(\rho) = \sum_{i=1}^n V_i \rho V_i^*$ be a minimal Kraus representation of the channel Φ. Then

$$\widehat{\Phi}(\rho) = \sum_{i,j=1}^n \mathrm{Tr} V_i \rho V_j^* |i\rangle\langle j| \qquad \text{and} \qquad \widehat{\Phi}^*(A) = \sum_{i,j=1}^n \langle j|A|i\rangle V_j^* V_i,$$

where $\{|i\rangle\}_{i=1}^n$ is an orthonormal basis in the n-dimensional Hilbert space $\mathcal{H}_E$.

Let $\{\pi_k, |\varphi_k\rangle\langle\varphi_k|\}$ be an optimal ensemble of pure states for the channel Φ with a full rank average state. We may assume that $\{|\varphi_k\rangle\}_{k=1}^m$, $m = \dim \mathcal{H}_A$, is a basis in the space $\mathcal{H}_A$. Let $\hat{\pi}_k = 1/m$, $k = \overline{1, m}$. Then $\hat{\rho} = \sum_{k=1}^m \hat{\pi}_k |\varphi_k\rangle\langle\varphi_k|$ is a full rank state in $\mathfrak{S}(\mathcal{H}_A)$. Since $\mathcal{H}_E$ is an environment space of minimal dimension, $\widehat{\Phi}(\hat{\rho})$ is a full rank state in $\mathfrak{S}(\mathcal{H}_E)$.

Let $|\phi_k\rangle = \sqrt{\hat{\pi}_k \hat{\rho}^{-1}}|\varphi_k\rangle$ and $B_k = \hat{\pi}_k [\widehat{\Phi}(\hat{\rho})]^{-1/2} \widehat{\Phi}(|\varphi_k\rangle\langle\varphi_k|)[\widehat{\Phi}(\hat{\rho})]^{-1/2}$, $k = \overline{1, m}$. Since $\sum_{k=1}^m |\phi_k\rangle\langle\phi_k| = I_{\mathcal{H}_A}$, $\{|\phi_k\rangle\}_{k=1}^m$ is an orthonormal basis in $\mathcal{H}_A$. By Theorem 1, B) $|\phi_k\rangle\langle\phi_k| = \widehat{\Phi}^*(B_k)$ for all k. By the spectral theorem $B_k = \sum_p |\psi_k^p\rangle\langle\psi_k^p|$, where $\{|\psi_k^p\rangle\}_p$ is a set of vectors in $\mathcal{H}_E$, for each k. Since $\widehat{\Phi}(\hat{\rho})$ is a full rank state, we have

$$\sum_{k,p} |\psi_k^p\rangle\langle\psi_k^p| = \sum_k B_k = I_E.$$

By Lemma 1 below $\Phi(\rho) = \sum_{k,p} W_{kp} \rho W_{kp}^*$, where $W_{kp} = \sum_{i=1}^n \langle\psi_k^p|i\rangle V_i$.

Since $|\phi_k\rangle\langle\phi_k| = \widehat{\Phi}^*\left(\sum_p |\psi_k^p\rangle\langle\psi_k^p|\right)$ for each k and

$$\widehat{\Phi}^*(|\psi_k^p\rangle\langle\psi_k^p|) = \sum_{i,j=1}^n \langle j|\psi_k^p\rangle\langle\psi_k^p|i\rangle V_j^* V_i = W_{kp}^* W_{kp},$$

10

there exists a collection $\{|\beta_{kp}\rangle\}$ of vectors in $\mathcal{H}_B$ such that $W_{kp} = |\beta_{kp}\rangle\langle\phi_k|$ and $\sum_p \|\beta_{kp}\|^2 = 1$ for each k. Hence

$$\Phi(\rho) = \sum_{k,p} W_{kp}\rho W_{kp}^* = \sum_k \langle\phi_k|\rho|\phi_k\rangle \sum_p |\beta_{kp}\rangle\langle\beta_{kp}|.$$

$\square$

Lemma 1. *Let* $\Phi(\rho) = \sum_{i=1}^n V_i\rho V_i^*$ *be a quantum channel and* $\{|i\rangle\}_{i=1}^n$ *be an orthonormal basis in the n-dimensional Hilbert space* $\mathcal{H}_E$. *An arbitrary overcomplete system* $\{|\psi_k\rangle\}_k$ *of vectors in* $\mathcal{H}_E$ *generates the Kraus representation* $\Phi(\rho) = \sum_k W_k\rho W_k^*$ *of the channel* Φ, *where* $W_k = \sum_{i=1}^n \langle\psi_k|i\rangle V_i$.

Proof. Since $\sum_k |\psi_k\rangle\langle\psi_k| = I_E$, we have

$$\sum_k W_k\rho W_k^* = \sum_{i,j=1}^n V_i\rho V_j^* \sum_k \langle\psi_k|i\rangle\langle j|\psi_k\rangle$$

$$= \sum_{i,j=1}^n V_i\rho V_j^* \sum_k \text{Tr}|i\rangle\langle j||\psi_k\rangle\langle\psi_k| \quad = \sum_{i=1}^n V_i\rho V_i^*.$$

$\square$

Remark 3. The assertions of Theorem 2 agree with the obtained in [9] criterion for the equality $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$ for the quantum-classical channel

$$\Phi(\rho) = \sum_k [\text{Tr}M_k\rho]|k\rangle\langle k|$$

defined by the collection $\{M_k\}$ of positive operators in $\mathcal{H}_A$ such that $\sum_k M_k = I_A$, where $\{|k\rangle\}$ is an orthonormal basis in $\mathcal{H}_B$. Indeed, it is easy to see that this channel is classical-quantum if and only if $M_k M_l = M_l M_k$ for all k, l.

Since $\mathcal{H}_\Phi^\chi = \mathcal{H}_A$ means existence of an optimal ensemble for the channel Φ with a full rank average state, Theorem 2 implies the following criterion for coincidence of the capacities.

Corollary 1. *Let* Φ *be a quantum channel for which there exists an optimal ensemble with a full rank average state. Then*

$$C_{\text{ea}}(\Phi) = \bar{C}(\Phi) \quad \Leftrightarrow \quad \Phi \text{ is a classical-quantum channel.}$$

11

The following example proposed in [3] (as an example of non–entanglement-breaking channel such that $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$) shows that the "full rank average state" condition in Corollary 1 is essential.

Example 2. Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be qubit spaces. Let $\{|k\rangle\}_{k=1}^4$ and $\{|-\rangle, |+\rangle\}$ be orthonormal bases in $\mathcal{K} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and in $\mathcal{H}_3$ correspondingly. Consider the channel

$$\Phi(\rho) = \sum_{k=1}^4 \left[\langle k| \otimes \langle +| \right] \rho \left[|k\rangle \otimes |+\rangle \right] |k\rangle\langle k| + \tfrac{1}{2} I_{\mathcal{H}_2} \otimes \text{Tr}_{\mathcal{H}_2 \otimes \mathcal{H}_3} [I_{\mathcal{K}} \otimes |-\rangle\langle -|] \rho$$

from $\mathfrak{S}(\mathcal{K} \otimes \mathcal{H}_3)$ into $\mathfrak{S}(\mathcal{K})$. It is easy to show that $C_{\text{ea}}(\Phi) = \bar{C}(\Phi) = 2$ and $Q(\Phi) = 1$ [3]. Thus the channel Φ is non-entanglement-breaking and hence it is not classical-quantum.

Since $\bar{C}(\Phi) = 2 = \log \dim \mathcal{K}$, any optimal ensemble for the channel Φ can not contain states with nonzero output entropy. Thus the subspace $\mathcal{H}_\Phi^\chi$ consists of vectors $|\varphi\rangle \otimes |+\rangle$, $|\varphi\rangle \in \mathcal{K}$. Hence the χ-essential part of the channel Φ is isomorphic to the classical-quantum channel $\rho \mapsto \sum_{k=1}^4 \langle k|\rho|k\rangle |k\rangle\langle k|$ (in accordance with Theorem 2, B)).

2.4 On covariant channels

The class of channels, for which the conditions of the parts B and C of Proposition 1 and of Corollary 1 hold simultaneously, contains any channel Φ *covariant* with respect to representations $\{V_g\}_{g \in G}$ and $\{W_g\}_{g \in G}$ of a compact group G in the sense that

$$\Phi(V_g \rho V_g^*) = W_g \Phi(\rho) W_g^*, \quad \forall g \in G, \tag{14}$$

provided the representation $\{V_g\}_{g \in G}$ is irreducible. Indeed, irreducibility of the representation $\{V_g\}_{g \in G}$ implies

$$\rho_c \doteq (\dim \mathcal{H}_A)^{-1} I_A = \int_G V_g \rho V_g^* \, \mu_H(dg), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}_A), \tag{15}$$

where μ_H is the Haar measure on the group G [11]. So, to prove that

$$\bar{C}(\Phi) = \chi_\Phi(\rho_c), \quad \bar{C}(\widehat{\Phi}) = \chi_{\widehat{\Phi}}(\rho_c), \quad C_{\text{ea}}(\Phi) = I(\rho_c, \Phi) \tag{16}$$

it is sufficient, by concavity of the χ-function and of the quantum mutual information, to show that

$$\chi_\Phi(\rho) = \chi_\Phi(V_g \rho V_g^*), \quad \chi_{\widehat\Phi}(\rho) = \chi_{\widehat\Phi}(V_g \rho V_g^*), \quad I(\rho, \Phi) = I(V_g \rho V_g^*, \Phi) \quad (17)$$

for all $g \in G$ and $\rho \in \mathfrak{S}(\mathcal{H}_A)$.

The first and the third equalities in (17) can be easily proved by using (3) and the well known expression for the quantum mutual information via the relative entropy (by means of invariance of the relative entropy with respect to unitary transformations of the both their arguments). By these equalities the second one follows from (6).

The class of covariant channels is sufficiently large, it contains all unital qubit channels and nontrivial classes of channels in higher dimensions [6, 11].

By using (15) and (16) it is easy to show that (cf.[11])

$$\bar{C}(\Phi) = H(\Phi(\rho_c)) - H_{\min}(\Phi), \qquad \bar{C}(\widehat\Phi) = H(\widehat\Phi(\rho_c)) - H_{\min}(\Phi),$$

$$(18)$$

$$C_{\mathrm{ea}}(\Phi) = \log \dim \mathcal{H}_A + H(\Phi(\rho_c)) - H(\widehat\Phi(\rho_c))$$

for any channel $\Phi \colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ satisfying the above covariance condition, where $H_{\min}(\Phi) = \min_{\rho \in \mathfrak{S}(\mathcal{H}_A)} H(\Phi(\rho))$ is the minimal output entropy of the channel Φ (coinciding with $H_{\min}(\widehat\Phi)$). If, in addition, the representation $\{W_g\}_{g \in G}$ is also irreducible then $H(\Phi(\rho_c))$ in (18) can be replaced by $\log \dim \mathcal{H}_B$ [11].

Let $Q_1(\Phi) = \max_{\rho \in \mathfrak{S}(\mathcal{H}_A)} I_c(\rho, \Phi)$ and $Q(\Phi) = \lim_{n \to +\infty} n^{-1} Q_1(\Phi^{\otimes n})$ be the quantum capacity of the channel Φ. By the above observations Proposition 1 and Corollary 1 imply the following assertions.

Proposition 2. *Let* $\Phi \colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *be a channel satisfying covariance condition* (14). *Then*

$$C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi) \quad \Leftrightarrow \quad \Phi \text{ is a classical-quantum channel.}$$

If, in addition, $\dim \mathcal{H}_B \geq \dim \mathcal{H}_A$ *and the representation* $\{W_g\}_{g \in G}$ *is irreducible then*

$$C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) = \log \dim \mathcal{H}_A - \bar{C}(\widehat\Phi) \leq Q_1(\Phi) + H_{\min}(\Phi),$$
$$C_{\mathrm{ea}}(\Phi) - C(\Phi) = \log \dim \mathcal{H}_A - C(\widehat\Phi) \leq Q(\Phi) + H_{\min}(\Phi).$$

13

Proof. If the representation $\{W_g\}_{g \in G}$ is irreducible then it is easy to show that $\Phi((\dim \mathcal{H}_A)^{-1} I_A) = (\dim \mathcal{H}_B)^{-1} I_B$ [11]. This and the condition $\dim \mathcal{H}_B \geq \dim \mathcal{H}_A$ imply $H(\Phi(\rho)) \geq H(\rho)$ for any $\rho \in \mathfrak{S}(\mathcal{H}_A)$ by monotonicity of the relative entropy. Coincidence of the last term in (9) and (10) with $H_{\min}(\Phi)$ follows from (15) and (16). $\qquad\square$

2.5   On degradable and anti-degradable channels

Expression (6) and the chain rule for the χ-function (i.e. $\chi_{\Psi \circ \Phi} \leq \chi_\Phi$) show that

$$C_{\mathrm{ea}}(\Phi_1) \leq \log \dim \mathcal{H}_A \leq C_{\mathrm{ea}}(\Phi_2) \qquad (19)$$

for any anti-degradable channel Φ_1 and any degradable channel Φ_2.[5] By using the Petz theorem [8, Theorem 3] one can show that if the first (correspondingly, the second) inequality in (19) holds with an equality then the anti-degradable channel Φ_1 is degradable (correspondingly, the degradable channel Φ_2 is anti-degradable).

The second inequality in (19) and Theorem 2 imply the following assertion.

Proposition 3. *If* $\Phi \colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *is a degradable channel then one of the following alternatives holds:*

- $\bar{C}(\Phi) < C_{\mathrm{ea}}(\Phi)$;

- Φ *is a classical-quantum channel having the representation*

$$\Phi(\rho) = \sum_{k=1}^{\dim \mathcal{H}_A} \langle k|\rho|k\rangle \sigma_k, \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \qquad (20)$$

  where $\{|k\rangle\}$ *is an orthonormal basis in* $\mathcal{H}_A$ *and* $\{\sigma_k\}$ *is a collection of states in* $\mathfrak{S}(\mathcal{H}_B)$ *with mutually orthogonal supports.*

Proof. Suppose that $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$. Since $\bar{C}(\Phi) \leq \log \dim \mathcal{H}_A$ for any channel Φ, the second inequality in (19) shows that $\bar{C}(\Phi) = \log \dim \mathcal{H}_A$ and hence the average state of any optimal ensemble for the channel Φ coincides with the chaotic state in $\mathfrak{S}(\mathcal{H}_A)$. By Corollary 1 Φ is a classical-quantum

---

[5]A channel Φ is called degradable if $\widehat{\Phi} = \Psi \circ \Phi$ for some channel Ψ, a channel Φ is called anti-degradable if $\widehat{\Phi}$ is a degradable channel [4].

channel having representation (20), in which $\{|k\rangle\}$ is an orthonormal basis in $\mathcal{H}_A$ and $\{\sigma_k\}$ is a collection of states in $\mathfrak{S}(\mathcal{H}_B)$. We will show that the supports of these states are mutually orthogonal.

Let $\sigma_k = \sum_{i=1}^{\dim \mathcal{H}_B} |\psi_{ki}\rangle\langle\psi_{ki}|$. Then $\Phi(\rho) = \sum_{k,i} W_{ki}\rho W_{ki}^*$, where $W_{ki} = |\psi_{ki}\rangle\langle k|$, and by using the standard representation for a complementary channel (cf. [12]) we obtain

$$\widehat{\Phi}(\rho) = \sum_{k,l=1}^{\dim \mathcal{H}_A} \langle k|\rho|l\rangle |k\rangle\langle l| \otimes \sum_{i,j=1}^{\dim \mathcal{H}_B} \langle \psi_{lj}|\psi_{ki}\rangle |i\rangle\langle j| \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B).$$

Since Φ is a degradable channel with representation (20), we have $\widehat{\Phi}(|k\rangle\langle l|) = \Psi \circ \Phi(|k\rangle\langle l|) = 0$ for all $k \neq l$. Hence the above expression for the channel $\widehat{\Phi}$ implies $\langle \psi_{lj}|\psi_{ki}\rangle = 0$ for all i,j and all $k \neq l$. It follows that $\operatorname{supp}\sigma_k \perp \operatorname{supp}\sigma_l$ for all $k \neq l$. $\qquad\square$

3   On channels with linear constraints

Speaking about different capacities of channels between finite dimensional quantum systems we can use any states for coding information. But dealing with real infinite dimensional channels we have to impose particular constraints on the choice of input code-states to avoid infinite values of the capacities and to be consistent with the physical implementation of the process of information transmission. A typical physically motivated constraint is defined by the requirement of bounded energy of states used for coding information. This constraint can be called linear, since it is determined by the linear inequality

$$\operatorname{Tr}H\rho \leq h, \quad h > 0, \tag{21}$$

where H is a positive operator – Hamiltonian of the input quantum system. Operational definitions of the Holevo capacity, the unassisted and the entanglement-assisted classical capacities of a quantum channel with linear constraints are given in [10], where the corresponding generalizations of the Holevo-Schumacher-Westmoreland and Bennett-Shor-Smolin-Thapliyal theorems are proved.

The aim of this section is to study relations between the above capacities of a quantum channel with linear constraints, in particular, to show that the question of coincidence of these capacities for a given channel depends on the form of the constraint.

For simplicity we restrict attention to the finite dimensional case.[6]

The Holevo capacity of the channel Φ with constraint (21) can be defined as follows

$$\bar{C}(\Phi, H, h) = \max_{\mathrm{Tr}H\rho \leq h} \chi_\Phi(\rho),$$

where χ_Φ is the χ-function of the channel Φ defined in (3). An ensemble $\{\pi_i, \rho_i\}$ of pure states with the average state $\bar{\rho}$ is called *optimal* for the channel Φ with constraint (21) if

$$\bar{C}(\Phi, H, h) = \chi_\Phi(\bar{\rho}) = \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\bar{\rho})) \qquad \text{and} \qquad \mathrm{Tr}H\bar{\rho} \leq h.$$

By the generalized Holevo-Schumacher-Westmoreland theorem [10, Proposition 3] the classical capacity of the channel Φ with constraint (21) can be expressed by the following regularization formula

$$C(\Phi, H, h) = \lim_{n\to+\infty} n^{-1}\bar{C}(\Phi^{\otimes n}, H_n, nh),$$

where $H_n = H \otimes I \otimes \ldots \otimes I + I \otimes H \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes H$ (each of n summands consists of n multiples).

By the generalized Bennett-Shor-Smolin-Thapliyal theorem [10, Proposition 4] the entanglement-assisted capacity of the channel Φ with constraint (21) is determined as follows

$$C_{\mathrm{ea}}(\Phi, H, h) = \max_{\mathrm{Tr}H\rho \leq h} I(\rho, \Phi),$$

where $I(\rho, \Phi)$ is the quantum mutual information of the channel Φ at the state ρ defined after (5).

Almost all the results of Section 2 concerning relations between the capacities $\bar{C}(\Phi)$ and $C_{\mathrm{ea}}(\Phi)$ can be reformulated for the corresponding capacities of a constrained channel. For example, instead of (8) we have

$$H(\rho_1) - \bar{C}(\widehat{\Phi}, H, h) \leq C_{\mathrm{ea}}(\Phi, H, h) - \bar{C}(\Phi, H, h)$$

$$\leq H(\rho_2) - \chi_{\widehat{\Phi}}(\rho_2) \underset{H(\Phi(\cdot))\geq H(\cdot)}{\leq} H(\Phi(\rho_2)) - \chi_{\widehat{\Phi}}(\rho_2) = I_c(\rho_2, \Phi) + \hat{H}_\Phi(\rho_2),$$

where ρ_1 and ρ_2 are states in $\mathfrak{S}(\mathcal{H}_A)$ such that $\mathrm{Tr}H\rho_i \leq h$, $i = 1, 2$, $\chi_\Phi(\rho_1) = \bar{C}(\Phi, H, h)$ and $I(\rho_2, \Phi) = C_{\mathrm{ea}}(\Phi, H, h)$.

---

[6]Generalizations to infinite dimensions are considered in the second part of [19].

By repeating the corresponding proofs it is easy to obtain the following proposition.

Proposition 4. *The assertions of Proposition 1, Theorem 1 and Theorem 2, B) remain valid with $\bar{C}(\Phi)$ and $C_{\mathrm{ea}}(\Phi)$ replaced respectively by $\bar{C}(\Phi, H, h)$ and $C_{\mathrm{ea}}(\Phi, H, h)$ (under the natural definition of the χ-essential part of the channel Φ with constraint (21)). The assertions of Theorem 2, A) remains valid under this replacement if the basis $\{|k\rangle\}$ in representation (13) of the channel Φ consists of eigenvectors of the operator H.*

The following example shows that the assertion of Theorem 2, A) without the additional condition is not valid for constrained channels.

Example 3. Consider the classical-quantum channel

$$\Pi(\rho) = \sum_k \langle k|\rho|k\rangle |k\rangle\langle k|,$$

where $\{|k\rangle\}$ is an orthonormal basis in $\mathcal{H}_A = \mathcal{H}_B$. Let $h < (\dim \mathcal{H}_A)^{-1}\mathrm{Tr}H$.

By using the generalized version of Theorem 1 we will show that

$$C_{\mathrm{ea}}(\Pi, H, h) = \bar{C}(\Pi, H, h)$$

if and only if the operator H is diagonizable in the basis $\{|k\rangle\}$.

Since $\Pi = \widehat{\Pi}$, we have $I(\rho, \Pi) = H(\rho)$ and $C_{\mathrm{ea}}(\Pi, H, h) = \max_{\mathrm{Tr}H\rho \leq h} H(\rho)$. By using the Lagrange method it is easy to show that the above maximum is attained at the unique state $\rho_* = (\mathrm{Tr}\exp(-\lambda H))^{-1}\exp(-\lambda H)$, where λ is determined by the equation $\mathrm{Tr}H\exp(-\lambda H) = h\mathrm{Tr}\exp(-\lambda H)$. If $C_{\mathrm{ea}}(\Pi, H, h) = \bar{C}(\Pi, H, h)$ then Theorem 1 implies existence of an ensemble $\{\pi_i, \rho_i\}$ of pure states with the average state ρ_* such that

$$\rho_i = \rho_*^{1/2}\Pi^*\left([\Pi(\rho_*)]^{-1/2}\Pi(\rho_i)[\Pi(\rho_*)]^{-1/2}\right)\rho_*^{1/2}, \quad \forall i.$$

Since $\Pi^* = \Pi$ and ρ_* is a full rank state, this equality may be valid only if $\rho_i = |k\rangle\langle k|$ for some k. Thus $\{|k\rangle\}$ is a basis of eigenvectors for the state ρ_* and hence for the operator H.

If the operator H is diagonizable in the basis $\{|k\rangle\}$ then $\rho_* = \sum_k \pi_k|k\rangle\langle k|$ and hence

$$\bar{C}(\Pi, H, h) \geq \sum_k \pi_k H(\Pi(|k\rangle\langle k|)\|\Pi(\rho_*)) = H(\rho_*) = C_{\mathrm{ea}}(\Pi, H, h).$$

17

Proposition 3 is generalized as follows.

Proposition 5. *Let* $\Phi \colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *be a degradable channel,* H *a positive operator,* $h > 0$ *and* $h_* = (\dim \mathcal{H}_A)^{-1}\mathrm{Tr}H$*. Then one of the following alternatives holds:*

- $\bar{C}(\Phi, H, h) < C_{\mathrm{ea}}(\Phi, H, h)$;

- Φ *is a classical-quantum channel having the representation*

$$\Phi(\rho) = \sum_{k=1}^{\dim \mathcal{H}_A} \langle k|\rho|k\rangle \sigma_k, \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \tag{22}$$

  where $\{\sigma_k\}$ *is a collection of states in* $\mathfrak{S}(\mathcal{H}_B)$ *with mutually orthogonal supports and* $\{|k\rangle\}$

    - is an orthonormal basis in $\mathcal{H}_A$*, if* $h \geq h_*$;

    - is the orthonormal basis of eigenvectors of the operator H*, if* $h < h_*$.

Proof. Since $\chi_\Phi(\rho) \leq H(\rho)$ and $I(\rho, \Phi) \geq H(\rho)$ (Φ is a degradable channel), the equality $\bar{C}(\Phi, H, h) = C_{\mathrm{ea}}(\Phi, H, h)$ may be valid only if

$$\bar{C}(\Phi, H, h) = C_{\mathrm{ea}}(\Phi, H, h) = \max_{\mathrm{Tr}H\rho \leq h} H(\rho).$$

If $h \geq h_*$ then this maximum coincides with $\log \dim \mathcal{H}_A$, which means that the constraint has no effect and hence the second alternative in Proposition 3 holds.

If $h < h_*$ then the above maximum is always attained at a full rank state and the generalized version of Theorem 2, B) implies that Φ is a classical-quantum channel having representation (22). Similar to the proof of Proposition 3 one can show that the states in the collection $\{\sigma_k\}$ have mutually orthogonal supports.

Show that the equality $\bar{C}(\Phi, H, h) = C_{\mathrm{ea}}(\Phi, H, h)$ may be valid in the case $h < h_*$ if and only if the operator H is diagonizable in the basis $\{|k\rangle\}$ from representation (22) of the channel Φ. For the channel $\Pi(\rho) = \sum_k \langle k|\rho|k\rangle |k\rangle\langle k|$ this assertion is proved in Example 3. To prove it in general case it suffices to note that $\bar{C}(\Phi, H, h) = \bar{C}(\Pi, H, h)$ and $C_{\mathrm{ea}}(\Phi, H, h) = C_{\mathrm{ea}}(\Pi, H, h)$. These equalities follow from the chain rules for the capacities, since it is easy to construct channels Ψ_1 and Ψ_2 such that $\Pi = \Psi_1 \circ \Phi$ and $\Phi = \Psi_2 \circ \Pi$. $\quad\square$

The following proposition shows that coincidence of $\bar{C}(\Phi, H, h)$ and $C_{\mathrm{ea}}(\Phi, H, h)$ for any constraint parameters (H, h) is a very strong requirement.

Proposition 6. *If* $\Phi \colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *is a quantum channel such that* $C_{\mathrm{ea}}(\Phi, H, h) = \bar{C}(\Phi, H, h)$ *for any operator* $H \geq 0$ *and* $h > 0$ *then* Φ *is a classical-quantum channel such that* $\chi_{\widehat{\Phi}}(\rho) = H(\rho)$ *for all* $\rho \in \mathfrak{S}(\mathcal{H}_A)$. *If the below Conjecture is true then* Φ *is the completely depolarizing channel.*

Proof. By Lemma 1 in [13] an arbitrary full rank state ρ in $\mathfrak{S}(\mathcal{H}_A)$ can be made the average state of an optimal ensemble for the channel Φ with constraint (21) by appropriate choice of the operator H. Hence the condition of the proposition and continuity arguments imply $I(\rho, \Phi) = \chi_{\Phi}(\rho)$ for any state ρ in $\mathfrak{S}(\mathcal{H}_A)$. By expression (6) this means that $\chi_{\widehat{\Phi}}(\rho) = H(\rho)$ for any state ρ in $\mathfrak{S}(\mathcal{H}_A)$. By the generalized version of Theorem 2, B) Φ is a classical-quantum channel. $\qquad\square$

Conjecture. *If* $\Phi \colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *is a quantum channel such that* $\chi_{\Phi}(\rho) = H(\rho)$ *for all* $\rho \in \mathfrak{S}(\mathcal{H}_A)$ *then the channel* Φ *coincides (up to unitary equivalence) with the channel* $\rho \mapsto \rho \otimes \sigma$ *for some state* σ.

4 The function $\Delta_{\Phi}(\rho) = I(\rho, \Phi) - \chi_{\Phi}(\rho)$ and its maximal value

Central role in analysis of relations between entanglement-assisted and unassisted classical capacities of a quantum channel Φ is played by the function

$$\Delta_{\Phi}(\rho) = I(\rho, \Phi) - \chi_{\Phi}(\rho)$$

introduced in Section 2, where it was mentioned that

$$\Delta_{\Phi}(\rho) = H(\rho) - \chi_{\widehat{\Phi}}(\rho) = \min_{\substack{\sum_i \pi_i \rho_i = \rho \\ \mathrm{rank}\rho_i = 1}} \sum_i \pi_i \left[H(\rho_i \| \rho) - H(\widehat{\Phi}(\rho_i) \| \widehat{\Phi}(\rho)) \right]$$

and that the above minimum is achieved at an ensemble $\{\pi_i, \rho_i\}$ of pure states if and only if this ensemble is χ_{Φ}-optimal in the sense of the following definition.

Definition 2. An ensemble $\{\pi_i, \rho_i\}$ of pure states is called χ_{Φ}-*optimal* if the maximum in definition (3) of the χ-function of the channel Φ is achieved at this ensemble.

Since $\hat{H}_\Phi \equiv \hat{H}_{\widehat{\Phi}}$, any χ_Φ-optimal ensemble is $\chi_{\widehat{\Phi}}$-optimal and vice versa.

The above formula for the function Δ_Φ and monotonicity of the relative entropy imply the following observation.

Lemma 2. *If Φ is a degradable channel then $\Delta_\Phi(\rho) \geq \Delta_{\widehat{\Phi}}(\rho)$ for all ρ.*

In the following theorem properties of the function Δ_Φ are described.

Theorem 3. *Let $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel and $\widehat{\Phi}\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_E)$ its complementary channel. Δ_Φ is a nonnegative continuous function on the set $\mathfrak{S}(\mathcal{H}_A)$ equal to zero on the subset $\operatorname{extr} \mathfrak{S}(\mathcal{H}_A)$ of pure states. It has the following properties:*

1) *if there exists a channel $\Theta\colon \mathfrak{S}(\mathcal{H}_E) \to \mathfrak{S}(\mathcal{H}_A)$ such that*

$$\Theta(\widehat{\Phi}(\rho_i)) = \rho_i, \quad \forall i, \tag{23}$$

   for some ensemble $\{\pi_i, \rho_i\}$ of pure states with the average state ρ then $\Delta_\Phi(\rho) = 0$ and the ensemble $\{\pi_i, \rho_i\}$ is χ_Φ-optimal;

2) *if $\Delta_\Phi(\rho) = 0$ then*

   - *(23) holds for any χ_Φ-optimal ensemble $\{\pi_i, \rho_i\}$ with the average state ρ, where Θ is a channel acting on a state σ supported by the subspace $\operatorname{supp}\widehat{\Phi}(\rho)$ as follows: $\Theta(\sigma) = A\widehat{\Phi}^*(B\sigma B)A$, $A = \rho^{1/2}$, $B = \widehat{\Phi}(\rho)^{-1/2}$;*

   - *$\Phi|_{\mathfrak{S}(\mathcal{H}_\rho)}$ is a classical-quantum subchannel of the channel Φ, where $\mathcal{H}_\rho$ is the support of the state ρ;*

   - *$\Delta_\Phi(\sum_i \lambda_i \rho_i) = 0$ for any χ_Φ-optimal ensemble $\{\pi_i, \rho_i\}$ with the average state ρ and any probability distribution $\{\lambda_i\}$.*

3) *the function Δ_Φ is concave on the set [7] $\left\{\sum_i \lambda_i \rho_i \mid \sum_i \lambda_i = 1,\ \lambda_i \geq 0\right\}$ for any χ_Φ-optimal ensemble $\{\pi_i, \rho_i\}$;*

4) monotonicity: *for an arbitrary channel $\Psi\colon \mathfrak{S}(\mathcal{H}_B) \to \mathfrak{S}(\mathcal{H}_C)$ the following inequality holds*

$$\Delta_{\Psi \circ \Phi}(\rho) \leq \Delta_\Phi(\rho), \quad \rho \in \mathfrak{S}(\mathcal{H}_A);$$

---

[7]The function Δ_Φ is not concave on $\mathfrak{S}(\mathcal{H}_A)$ in general, since otherwise we would obtain $\Delta_\Phi(\rho) \leq \Delta_\Phi(\rho_c) = 0$ for any covariant channel Φ such that $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$.

5) subadditivity for tensor product states: *for an arbitrary quantum channel* $\Psi \colon \mathfrak{S}(\mathcal{H}_C) \to \mathfrak{S}(\mathcal{H}_D)$ *the following inequality holds:*

$$\Delta_{\Phi \otimes \Psi}(\rho \otimes \sigma) \leq \Delta_{\Phi}(\rho) + \Delta_{\Psi}(\sigma), \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \quad \sigma \in \mathfrak{S}(\mathcal{H}_C),$$

which is satisfied with an equality if the strong additivity of the Holevo capacity holds for the channels Φ *and* Ψ *(see [13]).*

Proof. 1) This property follows from monotonicity of the relative entropy and the remark before Definition 2.

2) The first assertion follows from the Petz theorem [8, Theorem 3] characterizing the case in which monotonicity of the relative entropy holds with an equality.

The second assertion is derived from the first one by using the arguments from the proof of Theorem 2,B).

The third assertion follows from the first one and property 1).

3) Since $\hat{H}_{\Phi} \equiv \hat{H}_{\widehat{\Phi}}$, representation (4) for the function $\chi_{\widehat{\Phi}}$ implies

$$\Delta_{\Phi}(\rho) = \left[H(\rho) - H(\widehat{\Phi}(\rho)) \right] + \hat{H}_{\Phi}(\rho).$$

By the identity $H(\bar{\rho}) - \sum_i \pi_i H(\rho_i) = \sum_i \pi_i H(\rho_i \| \bar{\rho})$, where $\bar{\rho} = \sum_i \pi_i \rho_i$, concavity of the term in the square brackets on the set $\mathfrak{S}(\mathcal{H}_A)$ follows from monotonicity of the relative entropy. So, to prove this assertion it suffices to show that the function $\hat{H}_{\Phi}$ is affine on the set $\left\{ \sum_i \lambda_i \rho_i \mid \sum_i \lambda_i = 1, \, \lambda_i \geq 0 \right\}$. This can be done by noting that the function $\hat{H}_{\Phi}$ coincides with the double Fenchel transform of the function $H \circ \Phi$ and by using Proposition 1 in [1].

4) By using the Stinespring representation it is easy to show (see [4, the proof of Lemma 17]) that there exists a channel Θ such that $\widehat{\Phi} = \Theta \circ \widehat{\Psi \circ \Phi}$. Hence the chain rule for the χ-function implies

$$\Delta_{\Psi \circ \Phi}(\rho) = H(\rho) - \chi_{\widehat{\Psi \circ \Phi}}(\rho) \leq H(\rho) - \chi_{\widehat{\Phi}}(\rho) = \Delta_{\Phi}(\rho).$$

5) Since $\widehat{\Phi \otimes \Psi} = \widehat{\Phi} \otimes \widehat{\Psi}$ (see [12]), this assertion follows from the obvious inequality $\chi_{\widehat{\Phi} \otimes \widehat{\Psi}}(\rho \otimes \sigma) \geq \chi_{\widehat{\Phi}}(\rho) + \chi_{\widehat{\Psi}}(\sigma)$, which is satisfied with an equality if the strong additivity of the Holevo capacity holds for the channels Φ and Ψ [13]. $\square$

The following proposition shows the sense of the maximal value of the function Δ_Φ.

Proposition 7. *Let* $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *be a quantum channel. Then*

$$\max_{\rho\in\mathfrak{S}(\mathcal{H}_A)} \Delta_\Phi(\rho) = \sup_{H,h}\left[C_{\mathrm{ea}}(\Phi,H,h) - \bar{C}(\Phi,H,h)\right], \qquad (24)$$

where the supremum is over all pairs (positive operator $H \in \mathfrak{B}(\mathcal{H}_A)$, $h > 0$*).*

Proof. For given H and h let ρ be a state in $\mathfrak{S}(\mathcal{H}_A)$ such that $\mathrm{Tr}H\rho \leq h$ and $C_{\mathrm{ea}}(\Phi,H,h) = I(\rho,\Phi)$. Since $\bar{C}(\Phi,H,h) \geq \chi_\Phi(\rho)$, we have

$$\Delta_\Phi(\rho) = I(\rho,\Phi) - \chi_\Phi(\rho) \geq C_{\mathrm{ea}}(\Phi,H,h) - \bar{C}(\Phi,H,h),$$

This implies "$\geq$" in (24).

Let $\varepsilon > 0$ be arbitrary and ρ_ε be a full rank state in $\mathfrak{S}(\mathcal{H}_A)$ such that $\Delta_\Phi(\rho_\varepsilon) \geq \max_{\rho\in\mathfrak{S}(\mathcal{H}_A)} \Delta_\Phi(\rho) - \varepsilon$. By Lemma 1 in [13] there exists a pair (H,h) such that $\mathrm{Tr}H\rho_\varepsilon \leq h$ and $\bar{C}(\Phi,H,h) = \chi_\Phi(\rho_\varepsilon)$. Since $C_{\mathrm{ea}}(\Phi,H,h) \geq I(\rho_\varepsilon,\Phi)$, we have

$$C_{\mathrm{ea}}(\Phi,H,h) - \bar{C}(\Phi,H,h) \geq I(\rho_\varepsilon,\Phi) - \chi_\Phi(\rho_\varepsilon) = \Delta_\Phi(\rho_\varepsilon) \geq \max_{\rho\in\mathfrak{S}(\mathcal{H}_A)} \Delta_\Phi(\rho) - \varepsilon,$$

which implies "$\leq$" in (24). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

It is easy to see that $\max_{\rho\in\mathfrak{S}(\mathcal{H}_A)} \Delta_\Phi(\rho) \in [0, \log\dim\mathcal{H}_A]$. If $\Delta_\Phi(\rho) \equiv 0$ then the condition of Proposition 6 holds. If $\max_{\rho\in\mathfrak{S}(\mathcal{H}_A)} \Delta_\Phi(\rho) = \log\dim\mathcal{H}_A$ then Φ is unitary equivalent to the channel $\rho \mapsto \rho\otimes\sigma$, where σ is a given state. Indeed, this implies $\chi_{\widehat{\Phi}}(\rho_c) = 0$, where ρ_c is the chaotic state in $\mathfrak{S}(\mathcal{H}_A)$, and hence $\chi_{\widehat{\Phi}}(\rho) \equiv 0$ by concavity and nonnegativity of the χ-function, which means that $\widehat{\Phi}$ is a completely depolarizing channel.

Remark 4. Subadditivity of the function Δ_Φ (property 5 in Theorem 3) implies existence of the regularization $\Delta_\Phi^*(\rho) = \lim_{n\to+\infty} n^{-1}\Delta_{\Phi^{\otimes n}}(\rho^{\otimes n})$. By repeating the arguments from the proof of Proposition 7 and by using subadditivity of the quantum mutual information it is easy to show that

$$\max_{\rho\in\mathfrak{S}(\mathcal{H}_A)} \Delta_\Phi^*(\rho) \geq \sup_{H,h}\left[C_{\mathrm{ea}}(\Phi,H,h) - C(\Phi,H,h)\right].$$

The equality in this inequality is obvious if the strong additivity of the Holevo capacity holds for the channel Φ (see [13]), but it seems to be not valid in general.

Let $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ and $\Psi\colon \mathfrak{S}(\mathcal{H}_B) \to \mathfrak{S}(\mathcal{H}_C)$ be quantum channels. Monotonicity of the function Δ_Φ (property 4 in Theorem 3) shows that the inequality

$$C_{\mathrm{ea}}(\Psi \circ \Phi, H, h) - \bar{C}(\Psi \circ \Phi, H, h) \le C_{\mathrm{ea}}(\Phi, H, h) - \bar{C}(\Phi, H, h)$$

is valid if the functions $\rho \mapsto I(\rho, \Psi \circ \Phi)$ and $\rho \mapsto \chi_\Phi(\rho)$ have common maximum point under the condition $\mathrm{Tr} H\rho \le h$ (this holds for the unconstrained channels Φ and Ψ satisfying the covariance condition (14) with $\mathcal{H}_A = \mathcal{H}_B$ and $V_g = W_g$).

In general validity of the above inequality is an interesting open question, but monotonicity of the function Δ_Φ and Proposition 7 imply the following observation.

Corollary 2. *Let* $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *and* $\Psi\colon \mathfrak{S}(\mathcal{H}_B) \to \mathfrak{S}(\mathcal{H}_C)$ *be arbitrary quantum channels. Then*

$$\sup_{H,h} \left[C_{\mathrm{ea}}(\Psi \circ \Phi, H, h) - \bar{C}(\Psi \circ \Phi, H, h) \right] \le \sup_{H,h} \left[C_{\mathrm{ea}}(\Phi, H, h) - \bar{C}(\Phi, H, h) \right].$$

By introducing the parameter

$$D(\Phi) = \sup_{H,h} \left[C_{\mathrm{ea}}(\Phi, H, h) - \bar{C}(\Phi, H, h) \right]$$

of the channel $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ the above observations can be reformulated as follows:

- $D(\Phi) = \max_{\rho \in \mathfrak{S}(\mathcal{H}_A)} \Delta_\Phi(\rho)$;

- $D(\Psi \circ \Phi) \le D(\Phi)$ for any channel $\Psi\colon \mathfrak{S}(\mathcal{H}_B) \to \mathfrak{S}(\mathcal{H}_C)$;

- $D(\Phi) \in [0, \log \dim \mathcal{H}_A]$;

- $D(\Phi) = \log \dim \mathcal{H}_A$ if and only if the channel Φ is unitary equivalent to the noiseless channel $\rho \mapsto \rho \otimes \sigma$, where σ is a given state;

- $D(\Phi) = 0$ if Φ is a completely depolarizing channel ("if and only if" provided the Conjecture at the end of Section 3 is true).

The above properties show that the parameter $D(\Phi)$ can be considered as one of characteristics of the channel Φ describing its "level of noise".

Unfortunately, this parameter seems not to be easily calculated for nontrivial examples of quantum channels.

Generalizations of the results obtained in this paper to infinite dimensional constrained channels are presented in the second part of [19].

References

[1] K.M.R.Audenaert, S.L.Braunstein, "On Strong Superadditivity of the Entanglement of Formation", Comm. Math. Phys., 246:3, 443-452, 2004.

[2] C.H. Bennett, P.W. Shor, J.A.Smolin, A.V.Thapliyal "Entanglement-assisted classical capacity of noisy quantum channel", Phys. Rev. Lett. 83, 3081-3084, 1999; arXiv:quant-ph/9904023.

[3] C.H. Bennett, P.W. Shor, J.A.Smolin, A.V.Thapliyal "Entanglement-assisted capacity and the reverse Shannon theorem", IEEE Trans. Inform. Theory, 48:10, 2637-2655, 2002; arXiv:quant-ph/0106052.

[4] T.S.Cubitt, M.B.Ruskai, G.Smith "The structure of degradable quantum channels", J. Math. Phys. 49, 102104, 2008; arXiv:0802.1360.

[5] H.Fan "Remarks on entanglement assisted classical capacity", Phys. Lett. A, 313:3, 2003; arXiv:quant-ph/0301066.

[6] M.Fukuda, A.S.Holevo "On Weyl-covariant channels", arXiv:quant-ph/0510148.

[7] M.B.Hastings "Superadditivity of communication capacity using entangled inputs", Nature Physics, 5:255, 255-257, 2009; arXiv:0809.3972.

[8] P.Hayden, R.Jozsa, D. Petz, A.Winter "Structure of states which satisfy strong subadditivity of quantum entropy with equality", Commun. Math. Phys., 246:2, 359-374, 2004; arXiv:quant-ph/0304007.

[9] A.S.Holevo "Information capacity of quantum observable", Probl. Inf. Trans., 48:1, 1-10, 2012; arXiv:1103.2615.

[10] A.S.Holevo, "Classical capacities of quantum channels with constrained inputs", Probability Theory and Applications, 48:2, 359-374, 2003, arXiv quant-ph/0211170.

[11] A.S.Holevo "Remarks on the classical capacity of quantum channel", arXiv:quant-ph/0212025.

[12] A.S.Holevo "On complementary channels and the additivity problem", Probability Theory and Applications, 51:1, 134-143, 2006; arXiv:quant-ph/0509101.

[13] A.S.Holevo, M.E.Shirokov "On Shor's channel extension and constrained channels", Commun. Math. Phys. 249, 417-430, 2004; arXiv:quant-ph/0306196.

[14] M.Horodecki, P.W.Shor, M.B.Ruskai "General Entanglement Breaking Channels", Rev. Math. Phys 15, 629-641, 2003; arXiv:quant-ph/0302031.

[15] C.King, K.Matsumoto, M.Nathanson, M.B.Ruskai "Properties of Conjugate Channels with Applications to Additivity and Multiplicativity", Markov Process and Related Fields, V.13, P.391-423, 2007; arXiv:quant-ph/0509126.

[16] M.A.Nielsen, I.L.Chuang "Quantum Computation and Quantum Information", Cambridge University Press, 2000.

[17] B.Schumacher, M.D.Westmoreland "Optimal signal ensemble", arXiv:quant-ph/9912122.

[18] B.Schumacher, M.D.Westmoreland "Quantum privacy and quantum coherence", Phys. Rev. Lett. 80, 5695-5697, 1998; arXiv: quant-ph/9709058.

[19] M.E.Shirokov "A criterion for coincidence of the entanglement-assisted classical capacity and the Holevo capacity of a quantum channel", arXiv:1202.3449v2.